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Bound states of Dirac particle subjected to the pseudoscalar Hulthén potential

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Abstract
The (3+1)-dimensional Dirac equation is solved in the presence of the radial pseudoscalar Hulthén potential by using the usual approximation of the centrifugal potential. The approach proposed by Biedenharn for the Dirac–Coulomb problem is applied. Analytic bounded solutions of the Dirac equation with the pseudoscalar Hulthén potential are obtained in contrast to the pseudoscalar Coulomb one where there is no bounded solutions.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

During the last few years, the problem of the Dirac equation with pseudoscalar potentials (PSP) has been widely discussed either by solving the wave equation directly [1–5] or by using the path integral approach [6, 7]. Some considerable investigations have been made to understand the quantum behavior of Dirac particle subjected to a PSP; Villalba has shown that the inclusion of pseudoscalar potentials with a functional dependence inversely proportional to the distance in the (3+1) Dirac equation is not enough for obtaining bound states of the energy [1]. This conclusion has been communicated once again by the authors of [2]. However, de-Castro has discussed the existence of bounded states, in (1+1) dimension [3], in comparison with the case of (3+1) dimension. The pseudoscalar interactions are also analyzed in the context of the (1+1)-dimensional Dirac equation with non-Hermitian interactions but real energies. It is shown that the relevant hidden symmetry of the Dirac equation with such an interaction is pseudo-supersymmetry [4]. The energy bound states can then be found by analogy to the supersymmetric quantum mechanics. Elsewhere, we have proposed a straightforward method for solving the problem of Dirac particle subjected to a pseudoscalar potential by the supersymmetric path integrals, where we have described the spin degrees of freedom by odd Grassmannian variables [6]. This method proved most fruitful in finding analytical and exact expressions of the wavefunctions and the energy spectrum of the fermion in several cases, namely that the linear potential, the modified Pöschl–Teller potential and the Scarf II potential.

The aim of this paper is to searching for the complete set of bound state solutions of the (3+1)-dimensional Dirac equation with the pseudoscalar Hulthén potential

$$V_p(r) = -\alpha\delta \frac{\mathrm{e}^{-\delta r}}{1 - \mathrm{e}^{-\delta r}}, \tag{1}$$

where δ is the screening parameter. The spherically symmetric Hulthén potential has important applications in several areas of physical sciences such as nuclear and particle physics, atomic physics, condensed matter. For small r compared to $\frac{1}{\delta}$, the Hulthén potential approaches to Coulomb potential whereas for large r it approaches to zero exponentially that explains the use of this potential to study the screening effect.

As is known, the nonrelativistic Schrödinger equation with the Hulthén problem has an exact and analytic solution only for the s-waves (i.e. $l = 0$). However, in the case where the angular momentum is not zero, there are many investigations in searching approximate solutions [8–14]. Among the techniques used in the search for analytical solutions for any l states of the wave equation with this potential one is to approximate the centrifugal potential as

$$\frac{l(l+1)}{r^2} \approx l(l+1)\delta^2 \frac{\mathrm{e}^{-\delta r}}{(1 - \mathrm{e}^{-\delta r})^2}. \tag{2}$$

In order to find analytical solutions of the first-order Dirac equation we iterate it in the first stage to obtain a second-order equation. Next, by using the Biedenharn approach we diagonalize the angular operator so that the problem can be reduced to Schrödinger-like equation with a generalized Hulthén potential. Then we make an adequate change of the radial variable to obtain a second-order differential equation that resembles to Riemann-type equation and admits solutions in terms of hypergeometric series.

2. Solution of the Dirac equation

In a recent work, we have presented analytic solutions to the Klein–Gordon and Dirac equations with the ordinary Hulthén potential [15]. Our method is based on an approximation, already used in the Schrödinger equation, in which one replaces the centrifugal potential by an appropriate function leading to analytic solutions (i.e. with the use of the approximation in equation (2)). Let us proceed like in the previous work to study the motion of Dirac particle subjected to the pseudoscalar spherically symmetric Hulthén potential.

Before finding the bounded solutions of the relative Dirac equation, let us first recall that the Dirac equation with radial PSP has the following Hamiltonian form:

$$\mathrm{i}\frac{\partial}{\partial t}\Psi(t, \vec{r}) = \hat{H}\Psi(t, \vec{r}), \tag{3}$$

where

$$\hat{H} = \vec{\alpha}\vec{p} + \beta m + \beta\gamma^5 V_p(r), \tag{4}$$

with

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^5 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{5}$$

As is known, the general procedure used to solve the ordinary Dirac equation with spherically symmetric Lorentz vector potential is based on writing the spinor $\psi(\vec{r})$ as follows [16, 17]:

$$\psi(\vec{r}) = \frac{1}{r}\begin{pmatrix} F(r)\mathcal{Y}^M_{(j+\frac{1}{2})j} \\ \mathrm{i}G(r)\mathcal{Y}^M_{(j-\frac{1}{2})j} \end{pmatrix}, \tag{6}$$

with

$$\mathcal{Y}^M_{(j\pm\frac{1}{2})j} = \sum_{m,s} C^{(j\pm\frac{1}{2})\frac{1}{2}j}_{msM} Y^m_{j\pm\frac{1}{2}}(\theta,\phi)\chi^s \tag{7}$$

and

$$\frac{\vec{\sigma}\cdot\vec{r}}{r}\mathcal{Y}^M_{(j\pm\frac{1}{2})j} = -\mathcal{Y}^M_{(j\mp\frac{1}{2})j}. \tag{8}$$

The functions $F(r)$ and $G(r)$ then verify two coupled first-order equations.

This procedure, however, does not work for the case of the pseudoscalar potential due to the matrix γ^5 in (4). In such a case it is convenient to iterate the Dirac equation to obtain a quadratic equation. By setting $\Psi(t,\vec{r}) = \mathrm{e}^{\mathrm{i}Et}\psi(\vec{r})$, we obtain without difficulties

$$\left[E^2 - \vec{P}^2 - m^2 - V^2(r) - V'(r)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\vec{\sigma}\cdot\vec{r}}{r}\right]\psi(\vec{r}) = 0 \tag{9}$$

with

$$\vec{P}^2 = p_r^2 + \frac{L^2}{r^2} \tag{10}$$

and

$$p_r = -\mathrm{i}\frac{1}{r}\frac{\partial}{\partial r}r. \tag{11}$$

It is well known that the operator L is not a constant of motion and the operators L^2 and $\frac{\vec{\sigma}\cdot\vec{r}}{r}$ do not have the same eigenfunctions. In this case, we can apply the approach proposed by Biedenharn [18] and reviewed and applied to various physical problems [19–21].

Then by using the following properties of Pauli matrices and angular momentum

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\vec{a}\cdot\vec{b}) + \mathrm{i}\vec{\sigma}\cdot(\vec{a}\times\vec{b}) \tag{12}$$

and

$$\vec{L}\times\vec{L} = \mathrm{i}\vec{L}, \tag{13}$$

we get

$$L^2 = K^2 - K, \tag{14}$$

with

$$K = \vec{\sigma}\cdot\vec{L} + 1. \tag{15}$$

The analogous to the Biedenharn–Temple operator [18–21] is

$$\Lambda = \hat{A} + \hat{B}, \tag{16}$$

where the new operators $\hat{A}$ and $\hat{B}$ are defined as follows:

$$\hat{A} = \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix}, \qquad \hat{B} = -\alpha\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{r}}{r} & 0 \\ 0 & -\frac{\vec{\sigma}\cdot\vec{r}}{r} \end{pmatrix}. \tag{17}$$

It is easy to show that

$$\{\hat{A}, \hat{B}\} = 0, \tag{18}$$

and, consequently,

$$\Lambda^2 = \hat{A}^2 + \hat{B}^2 = K^2 + \alpha^2. \tag{19}$$

Then, according to the model of effective screened potential, by using the usual approximation of the centrifugal potential the quadratic wave equation will take the following stationary form:

$$\left[-p_r^2 + E^2 - m^2 + \alpha^2\delta^2 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - \Lambda(\Lambda-1)\delta^2 \frac{\mathrm{e}^{-\delta r}}{(1-\mathrm{e}^{-\delta r})^2}\right]\psi(\vec{r}) = 0. \tag{20}$$

Let λ be an eigenvalue of the operator Λ. Being aware of (19) we find

$$\lambda = \pm\sqrt{\left(j+\frac{1}{2}\right)^2 + \alpha^2}. \tag{21}$$

In order to separate the angular dependence of the wavefunction we assume that

$$\psi(\vec{r}) = \frac{1}{r}\left(\mathbb{Y}_j^+(\theta,\phi)\xi_+(r) + \mathbb{Y}_j^-(\theta,\phi)\xi_-(r)\right), \tag{22}$$

where the functions $\mathbb{Y}_j^\pm(\theta,\phi)$, which are the eigenfunctions of the operator Λ

$$\Lambda\mathbb{Y}_j^\pm(\theta,\phi) = \pm|\lambda|\mathbb{Y}_j^\pm(\theta,\phi), \tag{23}$$

are given by

$$\mathbb{Y}_j^+(\theta,\phi) = \begin{pmatrix} \mathrm{i}\alpha\mathcal{Y}^M_{(j+\frac{1}{2})j} - \left(j+\frac{1}{2}-|\lambda|\right)\mathcal{Y}^M_{(j-\frac{1}{2})j} \\ 0 \end{pmatrix} \tag{24}$$

and

$$\mathbb{Y}_j^-(\theta,\phi) = \begin{pmatrix} 0 \\ \mathrm{i}\alpha\mathcal{Y}^M_{(j-\frac{1}{2})j} - (j+\frac{1}{2}-|\lambda|)\mathcal{Y}^M_{(j+\frac{1}{2})j} \end{pmatrix}. \tag{25}$$

By substituting equation (22) into (20), we obtain two radial wave equations, respectively, for $\xi_+(r)$ and $\xi_-(r)$:

$$\left[\frac{\partial^2}{\partial r^2} + E^2 - m^2 + \alpha^2\delta^2 \frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} - \lambda_\pm(\lambda_\pm+1)\delta^2 \frac{\mathrm{e}^{-\delta r}}{(1-\mathrm{e}^{-\delta r})^2}\right]\xi_\pm(r) = 0, \tag{26}$$

where the parameters $\lambda_\pm$ are given by

$$\lambda_\pm = \sqrt{\left(j+\frac{1}{2}\right)^2 + \alpha^2} - \frac{1\pm 1}{2}. \tag{27}$$

Then, by making the change of variable from r to $y = \mathrm{e}^{-\delta r}$, which maps the interval $r \in]0, +\infty[$ into $y \in]0, 1[$ and by setting $\xi_\pm(r) \equiv f_\pm(y)$, and

$$k = \sqrt{E^2 - m^2} \qquad k' = \sqrt{E^2 - m^2 + \delta^2\alpha^2}, \tag{28}$$

we obtain the differential equation

$$\left[\frac{\partial^2}{\partial y^2} + \frac{1}{y}\frac{\partial}{\partial y} + \left(\frac{k^2}{\delta^2}\frac{1}{y} - \frac{k'^2}{\delta^2} - \lambda_\pm(\lambda_\pm+1)\frac{1}{1-y}\right)\frac{1}{y(1-y)}\right] f_\pm(y) = 0, \tag{29}$$

which resembles to Riemann-type equations [22]

$$\left[\frac{\partial^2}{\partial y^2}+\left(\frac{1-a-a'}{y}-\frac{1-c-c'}{1-y}\right)\frac{\partial}{\partial y}+\left(\frac{aa'}{y}-bb'+\frac{cc'}{(1-y)}\right)\frac{1}{y(1-y)}\right]f_\pm(y)=0 \tag{30}$$

where

$$a=-a'=\mathrm{i}\frac{k}{\delta}\qquad b=-b'=\mathrm{i}\frac{k'}{\delta}\qquad c=1-c'=\lambda_\pm+1 \tag{31}$$

with $a+a'+b+b'+c+c'=1$.

Thus, the associated solution can be expressed in terms of hypergeometric function

$$f_\pm(y)=y^a(1-y)^c\times\ {}_2F_1(u,v,w,y) \tag{32}$$

where

$$u=a+b+c\qquad v=a+b'+c\qquad w=2a+1. \tag{33}$$

Now, from the poles of the function $f_\pm(y)$ we easily determine the bound states. By putting $u=-n$, $(n=0,1,2,\ldots)$ we obtain, after some algebraic manipulations, the energy spectrum of the problem

$$E^2-m^2=-\frac{\delta^2}{4}\left(n+\lambda_\pm+1-\frac{\alpha^2}{n+\lambda_\pm+1}\right)^2. \tag{34}$$

The bounded solutions can be determined by the use of property [23]

$$\begin{aligned}{}_2F_1(u,v,w;y)&=\frac{\Gamma(w)\Gamma(w-u-v)}{\Gamma(w-u)\Gamma(w-v)}\,{}_2F_1(u,v,u+v-w+1;1-y)\\&\quad+\frac{\Gamma(w)\Gamma(u+v-w)}{\Gamma(u)\Gamma(v)}(1-y)^{w-u-v}\,{}_2F_1(w-u,w-v,w-u-v+1;1-y).\end{aligned} \tag{35}$$

One can get, easily, the analytic expression of the states $\xi_\pm(r)$

$$\xi_\pm(r)=C^\pm\,\mathrm{e}^{(n+\lambda_\pm-\mu+1)\frac{\delta}{2}r}(1-\mathrm{e}^{-\delta r})^{\lambda_\pm+1}F_1(-n,\mu+\lambda_\pm+1,2\lambda_\pm+2;1-\mathrm{e}^{-\delta r}) \tag{36}$$

where

$$\mu=\frac{\alpha^2}{n+\lambda_\pm+1} \tag{37}$$

and the normalization constants $C^\pm$ are given by

$$C^\pm=\sqrt{\frac{\delta}{2}}\frac{1}{\Gamma(2\lambda_\pm+2)}\left[\frac{\mu^2}{n+\lambda_\pm+1}-(n+\lambda_\pm+1)\right]^{\frac12}\left[\frac{\Gamma(\mu+\lambda_\pm+1)\Gamma(n+2\lambda_\pm+2)}{\Gamma(n+1)\Gamma(\mu-\lambda_\pm)}\right]^{\frac12}. \tag{38}$$

The functions $\xi_+(r)$ and $\xi_-(r)$ are related to one another by a first-order differential equation. In effect, using equation (equation (22) p 383 in [24])

$$\left(z(1-z)\frac{\partial}{\partial z}\right)F(\alpha,\beta,\gamma,z)=-(\gamma-\beta z-1)F(\alpha,\beta,\gamma,z)+(\gamma-1)F(\alpha-1,\beta,\gamma-1,z) \tag{39}$$

and the property (equation (5) p 1045 in [23])

$$F(\alpha-1,\beta,\gamma-1,y)=F(\alpha-1,\beta-1,\gamma-2,y)+\frac{(\alpha-1)(\gamma-\beta-1)}{(\gamma-1)(\gamma-2)}F(\alpha,\beta,\gamma,y) \tag{40}$$

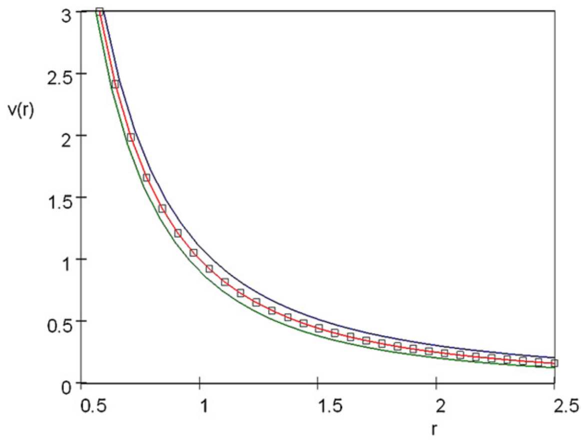

Figure 1. The red line represents the function $\frac{\delta^2\exp(-\delta r)}{(1-\exp(-\delta r))^2}$, the blue line represents the function $\frac{\delta^2}{(1-\exp(-\delta r))^2}$ and the green line represents the function $\frac{\delta^2\exp(-2\delta r)}{(1-\exp(-\delta r))^2}$. The boxes in black represent the centrifugal potential $1/r^2$. The parameter δ is taken $\delta = 0.1$.

we get

$$\frac{\mathrm{d}}{\mathrm{d}r} - \left(\delta\frac{\kappa + \varkappa\,\mathrm{e}^{-\delta r}}{(1-\mathrm{e}^{-\delta r})} + \frac{\delta\alpha^2}{2\lambda_-} + \mathcal{E}\right)\xi_{-,n}(z) = \delta(2\lambda_- + 1)\xi_{+,n+1}(z) \tag{41}$$

with

$$\begin{aligned} \frac{\kappa}{\delta} &= -\lambda_- + \mu - \frac{1}{2}(n+1) \\ \varkappa &= \frac{\lambda_-}{2} + n + 1. \end{aligned} \tag{42}$$

Note that the similar relation can be obtained for $\xi_{-,n+1}(z)$ and $\xi_{+,n}(z)$.

Let us give at the end of this section some concluding remarks.

The first one is that the approximation used in this work and in many papers [8]–[14] is a special case of the more general approximation

$$\frac{1}{r^2} \approx \delta^2\frac{\mathrm{e}^{-\mu\delta r}}{(1-\mathrm{e}^{-\delta r})^2}, \tag{43}$$

where μ is a dimensionless parameter and analytical solutions can be obtained for some values of the parameter μ (i.e. $\mu = 0, 1, 2$). The good approximation, however, is to take $\mu = 1$, as is shown in figure 1. (The blue line, the red line and the green one represent the function $\delta^2\frac{\exp(-\mu\delta r)}{(1-\exp(-\delta r))^2}$, respectively, with $\mu = 0, 1, 2$. The boxes in black represent the centrifugal potential $1/r^2$. The parameter δ is taken $\delta = 0.1$.).

The second remark is that in the limit $\delta \to 0$, the Hulthén potential becomes Coulomb one and equation (34) becomes $E^2 = m^2$ that explains the absence of the bound states for the Coulomb potential as it has been concluded in [1, 2].

We also remark that one can obtain the same energy spectrum of the electron and wavefunctions by using the path integral approach according to [25].

3. Conclusion

In this paper, we have analytically solved the Dirac equation in (3+1) dimensions with a radial pseudoscalar Hulthén potential. The separation of angular and radial parts was accomplished by iterating the first-order Dirac equation to obtain a Schrödinger-like second-order equation and by using the approach proposed by Biedenharn for the Dirac–Coulomb problem. The angular part becomes a linear combination of two-component spherical harmonics. For the radial part by using an appropriate approximation and by making a change of variable r, we obtain a second-order differential equation that resembles to Riemann-type equation and admits solutions in terms of hypergeometric series. The presented approach enables us to find approximate bounded solutions for any j-states of Dirac particle.

To conclude, we have succeeded to find the relativistic bound states for spin-half fermion interacting with the pseudoscalar potential in contrast to the case of pseudoscalar Coulomb potential, where there is no bounded solutions.

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